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## HYDRAULIC LINE DYNAMICS

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Fluid lines often play a major role in the dynamics of hydraulic control and other systems. The hydraulic line between two cross sections is characterized by a four terminal network with pressure and rate of flow the interacting variables. Use of this network leads to transcendental transfer functions that are not suited to the computation of system transients. The standard technique of power series expansions fails in that this yields instability in most applications where this instability does not actually occur. These difficulties are overcome by the use of infinite products. Only a few factors of these products are needed to compute transients to engineering precision. In contrast to the classical lumped constant approach to distributed systems the accuracy of the approximation can be seen from the factors directly. The technique applies to electrical transmission lines as well as hydraulic. By this method one can smooth transient responses to step changes arising in water hammer studies. Good agreement has been obtained between theory and experiment.

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### INTRODUCTION

High power and fast response hydraulic systems are required for many missile, aircraft and other applications. In the analysis and synthesis of such systems the fluid lines coupling the various components must be considered. The lumped constant approach is often employed, where 9 or 10 lumps per wavelength is used as a rule of thumb (1)<sup>1</sup>. This approach is limited since infinitely many degrees of freedom are actually involved. Where feasible the distributed parameter approach is to be preferred. The second order transfer matrix equation of electrical transmission line theory is used here to relate pressures and flows at two cross sections of a hydraulic line. The matrix equation describes a four terminal network, and agrees well with frequency response experiments for large and small pipes.

With the aid of boundary conditions one can often obtain transfer functions relating two of the four variables associated with two cross sections of a line. This is true, for example, if there is a fixed orifice at one of the sections, or there is a large reservoir at one section and a valve at the other discharging to atmosphere; or there might be a tank ahead of the valve. The transfer functions are transcendental in the Laplace variable  $s$ . It is convenient to employ these functions to compute frequency response but serious mathematical difficulties are encountered when they are used to calculate transient response. The standard technique of expanding the functions in power series yields characteristic equations with negative coefficients implying system instability where it does not actually occur. To overcome this difficulty the transfer functions are written here as quotients of in-

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1 Numbers in parentheses refer to the bibliography near the end of the paper.

finite products of factors linear in  $s$ . In practice one need keep only a few of the factors. A major advantage of this approach over the standard lumped constant technique is that one can see the accuracy of the approximation directly from the factors. The inclusion of more terms to approximate the transfer functions to greater bandwidth does not require solving successively higher degree algebraic equations. The infinite product approach applies to electric as well as hydraulic lines.

Notation

$a$	- Speed of sound in line ft/sec
$A$	- Cross sectional area $\text{ft}^2$
$A_1$	- Orifice area $\text{ft}^2$
$B$	- Constant in characteristic equation
$b_i(s)$	- Constant of integration, $i = 1, 2$
$E$	- Modulus of elasticity of pipe
$f$	- Wall thickness of pipe
$g$	- Acceleration of gravity $\text{ft/sec}^2$
$G(s)$	- Transfer matrix
$G_1(s)$	- Normalized transfer function between head and flow at section 1
$G_2(s)$	- Transfer function between heads at section 2 and section 1
$G_3(s)$	- Transfer function between flows at section 2 and section 1
$G_{1i}(s)$	- The $i$ th root factor approximation to $G_1(s)$
$h(x,t)$	- Pressure head deviation ft
$h_T$	- Pressure head deviation in chamber
$H(x,s)$	- Laplace transform of $h(x,t)$

$H_1(s)$	- Laplace transform of $h(x,t)$ at section 1
$j$	- $\sqrt{-1}$
$K$	- Bulk Modulus of fluid
$K_f$	- Pipe friction coefficient
$K_o$	- Reciprocal of slope of pressure versus flow curve at average flow
$L$	- Distance between sections 1 and 2, ft
$q_o$	- Flow increment $\text{ft}^3/\text{sec}$
$q_1(x,t)$	- Flow rate deviation $\text{ft}^3/\text{sec}$
$q_1(s)$	- Laplace transform of $q_1(t)$
$C(x,s)$	- Laplace transform of $A u(x,t)$
$r$	- Pipe inner radius
$s$	- Laplace variable, transformation with respect to time
$S(t)$	- Unit step function
$t$	- Time, sec
$T_e$	- $L/a$ , sec
$u(x,t)$	- Velocity of fluid in pipe, deviation, $\text{ft}/\text{sec}$
$U(x,s)$	- Laplace transform of $u(x,t)$
$V_1(s)$	- Column vector with elements $Q_1(s)$ and $H_1(s)$
$x$	- Axial pipe coordinate, ft
$X_B$	- Real part of root of characteristic equation
$z$	- Variable in characteristic equation
$z_o$	- $a/g A$ , $\text{sec}/\text{ft}^2$
$\alpha$	- Weight density of fluid divided by equivalent bulk modulus, $\text{ft}^{-1}$
$\beta(s)$	- $\sqrt{s^2 + gK_f s}$
$\delta$	- Damping factor

- $\mu$  - Viscosity of fluid, cp  
 $\omega$  - Frequency, radians/sec

### FUNDAMENTAL EQUATIONS

It is assumed in this analysis that the hydraulic line is a straight horizontal pipe of constant circular cross section. At each cross section average pressure head, velocity and fluid density are employed. Friction is first neglected. The coordinate of distance along the pipe and time are denoted by  $x$  and  $t$  respectively. The deviations in average velocity and pressure head at a cross section with coordinate  $x$  for the time  $t$  are given by  $u(x,t)$  and  $h(x,t)$  respectively. Letting  $\rho$  designate the fluid density,  $g$  the acceleration of gravity,  $K$  the bulk modulus of elasticity of the pipe material,  $f$  the pipe wall thickness and  $r$  the inner pipe radius, the well-known equations of flow are (1)

$$\frac{\partial u(x,t)}{\partial x} = -\alpha \frac{\partial h(x,t)}{\partial t} \quad [1]$$

$$\frac{\partial u(x,t)}{\partial t} = -g \frac{\partial h(x,t)}{\partial x} \quad [2]$$

where

$$\alpha = \rho g \left[ \frac{1}{K} + \frac{2r}{fE} \right]$$

The speed  $a$  of sound in the pipe is given by

$$a = \sqrt{\frac{g}{\alpha}} \quad [3]$$

Let  $U(x,s)$  and  $H(x,s)$  be the Laplace transforms of  $u(x,t)$  and  $h(x,t)$  respectively where  $s$  is the Laplace variable. Let  $q(x,t)$  be the average flow rate deviation at a pipe section of area  $A$ , whence

$$a(x,t) = A u(x,t) \quad [4]$$

Let  $Q(x,s)$  be the Laplace transform of  $q(x,t)$ . Let sections 1 and 2 designate the cross sections  $x = 0$  and  $x = L$  of the pipe. See Figure 1. The variables  $H_1(s)$ ,  $Q_1(s)$  for  $i = 1, 2$  are defined by

$$\begin{aligned} H_1(s) &= H(0,s) \\ H_2(s) &= H(L,s) \\ Q_1(s) &= Q(0,s) \\ Q_2(s) &= Q(L,s) \end{aligned} \quad [5]$$

The line impedance  $Z_0$  and time constant  $T_e$  are given by

$$\begin{aligned} Z_0 &= \frac{a}{Ag} \\ T_e &= \frac{L}{a} \end{aligned} \quad [6]$$

The initial conditions  $u(x,0^+) = h(x,0^+) = 0$  for flow rate and pressure head deviations at  $t = 0^+$  are assumed to hold. The solution of Equations [1] and [2] is now given by the matrix equation

$$G(s) V_1 = V_2 \quad [7]$$

where

$$G(s) = \begin{bmatrix} \cosh T_e s & -\frac{1}{Z_0} \sinh T_e s \\ -Z_0 \sinh T_e s & \cosh T_e s \end{bmatrix}$$

$$V_1 = \begin{bmatrix} Q_1(s) \\ H_1(s) \end{bmatrix}$$
$$V_2 = \begin{bmatrix} Q_2(s) \\ H_2(s) \end{bmatrix}$$

See the block diagram of Figure 2 where  $\Sigma$  is a summer.

Equation [7] applies if the pipe is not straight but has no sharp corners.

#### TESTS

The validity of Equation [7] was verified for large lines by frequency response runs of John Donelson and Rufus Oldenburger (8) at the Apalachia power house of the Tennessee Valley Authority. They oscillated the gates of a 53,500 horsepower hydraulic turbine and recorded hydraulic among other variables. This system involved an 8 mile 18 foot diameter tunnel, a differential surge tank, and two 600 foot penstocks, 11 feet in diameter. Excellent agreement between theory and practice was obtained over the frequency range of  $\frac{1}{2}$  cycle per hour to 2 cps. Mr. J. D. Regetz at the Lewis Center of the National Aeronautics and Space Administration made frequency response runs on a 1 inch diameter stainless steel pipe (3). The distance between the cross sections 1 and 2 of this pipe was 68 feet. Wall thickness was  $1/16$  in. Good agreement was obtained for 0.5 cps to 90 cps. The fluid was JP-4 jet fuel at 50 psi gauge, 25° C., average flow rate of 37 in<sup>3</sup>/sec, and Reynolds number about 14,000. The area of a valve near section 1 was varied. The fluid discharged to atmosphere at section 2. Pressures and flow rates at sections 1 and 2 were recorded. Runs on a  $\frac{1}{2}$  inch diameter line at the Automatic Control

Center of Purdue University have also been successful.

The tests mentioned above indicate that the basic water hammer equations [1] and [2] hold, friction effects are largely negligible (especially at high frequencies), and longitudinal pipe vibrations are small compared to the phenomena described by Equation [7].

#### LINE DISCHARGING THROUGH A FIXED ORIFICE

It is supposed that there is a fixed orifice at section 2 of Figure 1. This is the case of Figure 3 where the volume of the chamber C at section 2 is zero and the opening  $y(t)$  of the valve is constant. No restrictions are placed on the physical configuration to the left of section 1. Thus this is the case of a line discharging through an orifice. It is assumed that the following relation holds at section 2 for a constant  $K_o$  dependent on the orifice characteristics:

$$q(L,t) = K_o h(L,t) \quad [8]$$

Hence

$$Q_2(s) = K_o H_2(s) \quad [9]$$

Equation [8] holds for an arbitrary orifice and small changes in pressure and flow about a steady operating point. By Equations [7] and [9] line transfer functions  $G_1(s)$ ,  $G_2(s)$  and  $G_3(s)$  are obtained, where

$$G_1(s) = K_o \frac{H_1(s)}{Q_1(s)} = \frac{\cosh T_e s + K_o Z_o \sinh T_e s}{\cosh T_e s + \frac{1}{K_o Z_o} \sinh T_e s} \quad [10]$$

$$G_2(s) = \frac{H_2(s)}{H_1(s)} = \frac{1}{\cosh T_e s + K_o Z_o \sinh T_e s} \quad [11]$$

$$G_3(s) = \frac{Q_2(s)}{Q_1(s)} = \frac{1}{\cosh T_e s + \frac{1}{K_o Z_o} \sinh T_e s} \quad [12]$$

With  $s = j\omega$  for  $j = \sqrt{-1}$  Equation [10] yields the normalized line impedance  $G_1(j\omega)$ . To verify Equation [7] J. D. Regetz made theoretical and experimental plots of the magnitude and phase of this transfer function versus frequency.

#### LINE WITH CONSTANT PRESSURE AT ONE END

Consider the configuration of Figure 3 where there is a chamber C at section 2 of the line. The chamber pressure head deviation from equilibrium is denoted by  $h_T$ . The fluid discharges to atmosphere from the chamber C through a valve. The deviation in the effective orifice area of the valve is taken to be  $A_1 y(t)$  for a constant  $A_1$  and valve stroke deviation  $y(t)$ . Now

$$h_T = h(L,t) \quad [13]$$

For constants  $c_1$  and  $c_2$  depending on the characteristics of the valve, and a constant  $c_3$  depending on the bulk modulus of the fluid and chamber characteristics one has

$$q(L,t) = c_1 y(t) + c_2 h(L,t) + c_3 \frac{dh(L,t)}{dt} \quad [14]$$

Thus

$$Q_2(s) = c_1 Y(s) + c_2 H_2(s) + c_3 s H_2(s) \quad [15]$$

where  $Y(s)$  is the Laplace transform of  $y(t)$ .

It is assumed that there is a source of constant pressure at section 1, as when there is a large reservoir at this section, or accumulator supplying fluid, or a pump with a relief valve. Now

$$h(0,t) = 0 \quad [16]$$

whence

$$H_1(s) = 0 \quad [17]$$

By Equations [7] and [17]

$$Q_2(s) = Q_1(s) \cosh T_e s \quad [18]$$

$$H_2(s) = -Z_0 Q_1(s) \sinh T_e s \quad [19]$$

Equations [15], [18] and [19] imply that

$$\frac{Q_2(s)}{Y(s)} = c_1 \frac{\cosh T_e s}{\cosh T_e s + Z_0(c_2 + c_3 s) \sinh T_e s} \quad [20]$$

$$\frac{H_2(s)}{Y(s)} = -Z_0 c_1 \frac{\sinh T_e s}{\cosh T_e s + Z_0(c_2 + c_3 s) \sinh T_e s} \quad [21]$$

The case of a constant pressure source at section 1 and a valve at section 2 discharging to atmosphere occurs when the volume of chamber C is zero, whence Equations [20] and [21] apply with  $c_3 = 0$ .

FRICTION

Linear pipe friction may be included by using

$$\frac{\partial u(x,t)}{\partial t} = -g \left[ \frac{\partial h(x,t)}{\partial t} + K_f u(x,t) \right] \quad [22]$$

in place of Equation [2], where  $K_f$  is a friction constant. Equation [22] proved adequate to describe flow in  $\frac{1}{2}$  inch diameter lines. Let  $\beta(s)$  be defined by

$$\beta(s) = \sqrt{s^2 + g K_f s} \quad [23]$$

With the initial conditions

$$u(x,0^+) = h(x,0^+) = 0,$$

we have

$$G_{\beta(s)} v_1 = v_2 \quad [24]$$

where

$$G_{\beta(s)} = \begin{bmatrix} \cosh T_e \beta(s) & -\frac{1}{Z_o} \frac{s}{\beta(s)} \sinh T_e \beta(s) \\ -Z_o \frac{\beta(s)}{s} \sinh T_e \beta(s) & \cosh T_e \beta(s) \end{bmatrix}$$

With the fixed orifice as the boundary condition at section 2, the ratio of pressure head to flow deviations at section 1 is given by

$$\frac{H_1(s)}{Q_1(s)} = \frac{1}{K_o} \frac{\cosh T_e \beta(s) + K_o Z_o \frac{\beta(s)}{s} \sinh T_e \beta(s)}{\cosh T_e \beta(s) + \frac{1}{K_o Z_o} \frac{s}{\beta(s)} \sinh T_e \beta(s)} \quad [25]$$

For  $s$  numerically large  $\beta(s)$  may be replaced by  $s$  whence Equation [25]

reduces to Equation [10]. Since for frequency response  $s = j\omega$  it follows that friction effects diminish as the frequency increases, and are negligible at high frequencies.

For the tests of J. D. Regetz

$$Z_o = 28,700 \text{ sec/ft}^2$$

$$K_o = 5.43 \times 10^{-5} \text{ ft}^2/\text{sec}$$

$$T_e = 0.0176 \text{ sec}$$

$$K_f = 0.0264 \text{ sec/ft}$$

The differences in magnitude and phase angle for the input line impedances  $\{H_1(j\omega) / Q_1(j\omega)\}$  given by Equations [10] and [25] were less than 4% at 1 cps and less than 1% at 5 cps.

#### PERFECT TRANSMISSION

For the tests of J. D. Regetz

$$K_o Z_o = 1.56 \quad [26]$$

From Equation [10] the magnitude ratio of pressure head deviation to flow rate deviation at section 1 normally varies with the frequency. If, however,

$$K_o Z_o = 1 \quad [27]$$

the line transfer functions  $G_1(s)$ ,  $G_2(s)$  and  $G_3(s)$  become

$$G_1(s) = 1, \quad G_2(s) = G_3(s) = e^{-T_e s} \quad [28]$$

Where Condition [27] is satisfied, the analysis of the line is simple since pressure and flow rate deviations are proportional and in phase with each other at each cross section of the line. The proportion is independent of

frequency. Pressure and flow disturbances are propagated along the line as pure delays with the delay time  $T_e$  between sections 1 and 2. The authors feel that this phenomenon may have worthwhile practical applications.

#### ROOT FACTOR APPROXIMATIONS

The transfer functions in Equations [10] - [12], [20], [21] are all proportional to quotients of transcendental functions of the form  $F(z)$ , where

$$F(z) = \cosh z + B \sinh z \quad [29]$$

and  $z = T_e s$ . Here  $B$  is a constant or function of  $s$ . The same is true of the transfer functions arising from Equation [24] for the case of line friction, except that  $z$  is a more complicated function of  $s$ . Severe mathematical difficulties arise when the transcendental functions are employed directly to compute system response to step and other disturbances. When  $F(z)$ ,  $z = T_e s$ , is the denominator of such a transfer function the technique of expanding  $F(z)$  into a power series in  $z$ , and keeping lower order terms to obtain rational approximations to the transfer functions fails. Thus, keeping terms to the fifth degree yields the approximation  $F_5(z)$ , where

$$F_5(z) = 1 + Bz + \frac{z^2}{2!} + \frac{Bz^3}{3!} + \frac{z^4}{4!} + \frac{Bz^5}{5!} \quad [30]$$

For  $B \neq 0$  the function  $F_5(z)$  has a zero in the right half plane. It follows that fifth and higher degree approximations to  $F(z)$  yield instability where it does not occur physically. To avoid this difficulty the authors expand  $F(z)$  into an infinite product instead.

Let  $x$  and  $y$  be the real and imaginary parts of  $z$ , so that  $z = x + jy$ .

Writing  $\cosh z$  and  $\sinh z$  as

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

the equation

$$\cosh z + B \sinh z = 0 \quad [31]$$

becomes

$$e^{2x} \cos 2y + j e^{2x} \sin 2y = \frac{B - 1}{B + 1} \quad [32]$$

If  $B$  is real Equation [32] gives

$$e^{2x} \cos 2y = \frac{B - 1}{B + 1} \quad [33]$$

$$e^{2x} \sin 2y = 0 \quad [34]$$

Solving Equations [33] and [34] the roots of Equation [31] are found to be

$$\begin{aligned} z &= \frac{1}{2} \ln \left| \frac{B - 1}{B + 1} \right| \pm j \frac{2n + 1}{2} \pi & B < 1 \\ z &= \frac{1}{2} \ln \left| \frac{B - 1}{B + 1} \right| \pm j n \pi & B > 1 \end{aligned} \quad [35]$$

For  $B = 1$  there are no bounded roots of Equation [31].

Introduce  $x_B$  where

$$x_B = \frac{1}{2} \ln \frac{B - 1}{B + 1}$$

From the roots of Equation [31] the function  $F(z)$  can be factored into an infinite produce, as follows:

$$\cosh z + B \sinh z = \prod_{n=0}^{\infty} \left[ 1 - \frac{2X_B z - z^2}{X_B^2 + \left(\frac{2n+1}{2}\right)^2 \pi^2} \right] \quad B < 1 \quad [36]$$

$$\cosh z + B \sinh z = \left[ 1 - \frac{z}{X_B} \right] \prod_{n=1}^{\infty} \left[ 1 - \frac{2X_B z - z^2}{X_B^2 + n^2 \pi^2} \right] \quad B > 1 \quad [37]$$

A complex plane plot of the roots of Equation [31] is shown in Figure 4. A plot of  $X_B$  versus  $B$  is given in Figure 5.

The transfer functions  $G_1(s)$ ,  $G_2(s)$ ,  $G_3(s)$ ,  $Q_1(s) / Y(s)$  and  $H_2(s) / Y(s)$  for the cases of an orifice or valve at section 2 can be expressed as infinite products by Equations [36] and [37]. The root factor method is understood to be the procedure of finding the zeros of  $\cosh z + B \sinh z$  and expressing this function as a product of corresponding factors. The transfer function  $G_1(s)$  will be used to demonstrate the root factor method.

Let  $K_0 Z_0 = 1.56$  as for the Regetz experiments. The function  $G_1(s)$  is now given by

$$G_1(s) = \frac{\cosh T_e s + 1.56 \sinh T_e s}{\cosh T_e s + 0.642 \sinh T_e s} \quad [38]$$

The numerator of  $G_1(s)$  is given by the right side of Equation [37] with  $z = T_e s$  and  $B = 1.56$ , and the denominator by the right side of Equation [36] with  $z = T_e s$  and  $B = 0.642$ . It follows that

$$G_1(s) = \frac{\left[1 - \frac{T_e s}{X_B}\right] \prod_{n=1}^{\infty} \left[1 - \frac{2X_B T_e s - (T_e s)^2}{X_B + n^2 \pi^2}\right]}{\prod_{n=0}^{\infty} \left[1 - \frac{2X_B T_e s - (T_e s)^2}{X_B^2 + \left(\frac{2n+1}{2}\right)^2 \pi^2}\right]} \quad [39]$$

By Figure 5

$$X_B = -0.763$$

In the Regetz experiments  $T_e = 0.0176$ . The function  $G_1(s)$  is now

$$G_1(s) = \frac{\left[1 + \frac{s}{\omega_0}\right] \prod_{n=1}^{\infty} \left[1 + 2 \frac{\zeta_{1n} s}{\omega_{1n}} + \frac{s^2}{\omega_{2n}^2}\right]}{\prod_{n=0}^{\infty} \left[1 + 2 \frac{\zeta_{2n} s}{\omega_{2n}} + \frac{s^2}{\omega_{2n}^2}\right]} \quad [40]$$

where

$$\omega_0 = 43.4 \text{ radians/sec}$$

$$\omega_{1n} = 56.8 \sqrt{(0.763)^2 + (n \pi)^2}$$

$$\omega_{2n} = 56.8 \sqrt{(0.763)^2 + \left(\frac{2n+1}{2}\right)^2 \pi^2}$$

$$\zeta_{1n} = \sqrt{\frac{(0.763)^2}{(0.763)^2 + n^2 \pi^2}}$$

$$\zeta_{2n} = \sqrt{\frac{(0.763)^2}{(0.763)^2 + \left(\frac{2n+1}{2}\right)^2 \pi^2}}$$

For 0 to 90 cps the function  $G_1(j\omega)$  is approximated within 1 DB in magnitude and  $5^\circ$  in angle by taking  $n = 0$  to 5 in Equation [40], by letting  $s = j\omega$ , and neglecting all other terms in the infinite product. The approximation obtained by dropping the factors for  $n > m$  will be denoted by  $G_{1m}(j\omega)$ . In Figures 6a and b are shown the magnitude ratio and phase curves for the precise transfer function  $G_1(j\omega)$  and the approximation  $G_{15}(j\omega)$ .

Dropping the  $n \geq 2$  factors in Formula [40] yields  $G_{11}(s)$  where  $G_{11}(s)$  is a cubic in  $s$  divided by a quartic. There is excellent agreement between  $G_1(j\omega)$  and  $G_{11}(j\omega)$  from zero to 50 cps. For the sake of brevity the frequency response plots are omitted.

Dropping the  $n \geq 1$  factors in Formula [40], there results

$$G_{10}(s) = \frac{1 + \frac{s}{43.4}}{1 + 0.874 \left(\frac{s}{99.3}\right) + \left(\frac{s}{99.3}\right)^2} \quad [41]$$

In Figures 7a and 7b are plotted the frequency response curves for  $G_1(j\omega)$  and  $G_{10}(j\omega)$ . Clearly,  $G_{10}(j\omega)$  is a good approximation to  $G_1(j\omega)$  for 0 to 15 cps.

Thus the infinite products in Equation [40] converge rapidly. The root factor method yields accurate rational approximations to  $G_1(s)$ . The bandwidth of the other components in a system with hydraulic lines determines the largest value of  $\omega_{in}$  to include in the approximations. The largest value of the line frequency constant  $\omega_{in}$  should be about 1.5 times the bandwidth of the other transfer functions in the loop.

The numerator of the fraction on the right in Formula [20] for  $Q_2(s) / Y(s)$  is the single term  $\cosh T_0 s$ . Since Formula [31] holds when  $B = 0$  there

is no difficulty. On the other hand, the numerator  $\sinh T_e s$  of  $H_2(s) / Y(s)$  in Equation [21] corresponds to  $B = \infty$  where Equation [31] breaks down. To avoid this difficulty let

$$\left. \frac{\sinh T_e s}{T_e s} \right|_{s=0} = 1 \quad [42]$$

The Equation

$$\frac{\sinh T_e s}{s} = 0 \quad [43]$$

has the roots

$$T_e s = \pm j n \pi, \quad n \neq 0$$

which follows from Equation [35] when  $B \rightarrow \infty$ . Thus one may use

$$\sinh T_e s = T_e s \prod_{n=1}^{\infty} \left[ 1 + \frac{T_e^2 s^2}{n^2 \pi^2} \right] \quad [44]$$

In the denominators of Formulas [20] and [21]

$$B = Z_0 (c_2 + c_3 s) \quad [45]$$

In the chamber and valve case  $B$  is not a constant. It is a simple matter to solve Equation [31] with  $z = T_e s$  for given numerical values of  $Z_0$ ,  $c_2$  and  $c_3$ . Thus if

$$\begin{aligned} T_e &= 0.0176, \quad Z_0 = 28,700, \quad c_2 = 2.72 \times 10^{-5}, \\ c_3 &= 6.98 \times 10^{-6} T_e \end{aligned} \quad [46]$$

the roots of Equation [31] are

$$z = -0.685 \pm 1.22 j, -0.235 \pm 6.82 j, \\ -0.145 \pm 9.82 j, -0.095 \pm 12.0 j, \dots$$

whence

$$\cosh z + Z_0 \left( c_2 + \frac{c_3}{T_e} z \right) \sinh z = \left[ 1 + \frac{z^2 + 1.37z}{1.97} \right] \\ \times \left[ 1 + \frac{z^2 + 0.47z}{4.65} \right] \left[ 1 + \frac{z^2 + 0.29z}{97} \right] \left[ 1 + \frac{z^2 + 0.19z}{166} \right] \dots \quad [47]$$

Similarly more complicated functions B of s may be treated.

#### TRANSIENT RESPONSE

The pressure head transient  $h_1(t)$  at section 1 for a step change  $q_0$  in flow rate at section 1 will be obtained for the case of an orifice at section 2. By Equation [10] the ratio  $H_1(s) / Q_1(s)$  of the Laplace transforms of pressure head and flow rate at  $x = 0$  satisfies

$$\frac{H_1(s)}{Q_1(s)} = \frac{1}{K_0} G_1(s) \quad [48]$$

The step change  $q_0$  in flow rate corresponds to

$$Q_1(s) = \frac{q_0}{s}$$

It follows that

$$H_1(s) = \frac{q_o Z_o}{s} \frac{1 + \frac{1 - K_o Z_o}{1 + K_o Z_o} e^{-2T_e s}}{1 + \frac{K_o Z_o - 1}{K_o Z_o + 1} e^{-2T_e s}} \quad [49]$$

Division yields

$$H_1(s) = q_o Z_o \sum_{n=0}^{\infty} (-1)^n \mathcal{E}_n \left( \frac{K_o Z_o - 1}{K_o Z_o + 1} \right) \frac{e^{-2nT_e s}}{s} \quad [50]$$

where

$$\mathcal{E}_n = \begin{matrix} 1 & n = 0 \\ 2 & n \neq 0 \end{matrix}$$

Taking the inverse Laplace transform the precise response is found to be

$$h_1(t) = q_o Z_o \sum_{n=0}^{\infty} (-1)^n \mathcal{E}_n \left( \frac{K_o Z_o - 1}{K_o Z_o + 1} \right)^n S(t - 2nT_e) \quad [51]$$

where  $S(t)$  is the unit step function given by

$$S(t) = \begin{matrix} 0 & t \leq 0 \\ 1 & t > 0 \end{matrix}$$

Let  $h_{10}(t)$  denote the approximate response at section 1 computed by using  $G_{10}(s)$  of Equation [41] in place of  $G_1(s)$ . The corresponding approximate Laplace transform  $H_{10}(s)$  is then given by

$$H_{10}(s) = \frac{q_0}{sK_0} \frac{1 + \frac{s}{43.2}}{1 + 0.874 \left( \frac{s}{99.3} \right) + \left( \frac{s}{99.3} \right)^2} \quad [52]$$

whence

$$h_{10}(\tau) = \frac{q_0}{K_0} \left[ 1 - e^{-1.52\tau} (2.28) \cos (3.15 \tau + 1.12) \right] \quad [53]$$

where

$$\tau = \frac{t}{2T_0}$$

Curves of  $h_1(t)$  and  $h_{10}(t)$  versus multiples of  $T_0$  are plotted in Figure 8. The use of  $G_{10}(s)$  in place of  $G_1(s)$  yields a good fit to the actual transient (marked "theoretical"), and is to be preferred to it in that the  $h_{10}(t)$  curve is a smoothed version of the  $h_1(t)$  curve with the sharp corners removed. The smoothed solution  $h_{10}(t)$  is easier to use in analysis and synthesis.

The root factor method is valid for the computation of transients for systems with boundary conditions other than those treated in this paper. It can be applied with equal facility to the transfer functions of Paynter and Ezekial (6), Zweig (7) and others where a linear boundary condition is used.

#### OPERATIONS

Let

$$D = \frac{d}{dt}$$

By definition

$$(\cosh T_e D) f(t) = \frac{f(t + T_e) + f(t - T_e)}{2} \quad [54]$$

$$(\sinh T_e D) f(t) = \frac{f(t + T_e) - f(t - T_e)}{2} \quad [55]$$

The partial differential Equations [1] and [2] may be replaced by the ordinary differential equations

$$q(L,t) = (\cosh T_e D) q(0,t) - \frac{1}{Z_0} (\sinh T_e D) h(0,t) \quad [56]$$

$$h(L,t) = -Z_0 (\sinh T_e D) q(0,t) + (\cosh T_e D) h(0,t) \quad [57]$$

relating sections 1 and 2. Equation [48] may be taken as

$$h(0,t) = \frac{1}{K_0} \frac{\cosh T_e D + K_0 Z_0 \sinh T_e D}{\cosh T_e D + \frac{1}{K_0 Z_0} \sinh T_e D} q(0,t) \quad [58]$$

By Equation [40], dropping the  $n \geq 1$  terms,

$$h(0,t) = \frac{1}{K_0} \frac{1 + \frac{D}{\omega_0}}{1 + 2 \frac{\omega_{20}}{\omega_{20}} D + \frac{D^2}{\omega_{20}^2}} q(0,t) \quad [59]$$

The simple rational D-operator in Equation [59] suffices for first approx-

imation studies. The operators  $G_{10}(D)$  or  $G_{11}(D)$  are adequate unless the system variables change rapidly.

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ADDENDUM

The transfer matrix of a fluid line which applies when viscous effects are significant and may not be accounted for by a linear friction term included in the momentum equation of water hammer is given below.

$$G(s) = \begin{bmatrix} \cosh M(s) & -\frac{1}{Z(s)} \sinh M(s) \\ -Z(s) \sinh M(s) & \cosh M(s) \end{bmatrix} \quad (A)$$

where

$$M(s) = \frac{s L}{a N(s)}$$

$$Z(s) = \frac{a}{A_0 N(s)}$$

$$N(s) = \left[ \frac{-J_2(j\sqrt{\frac{a}{\gamma}} r_0)}{J_0(j\sqrt{\frac{a}{\gamma}} r_0)} \right]^{\frac{1}{2}}$$

where  $\rho$  is the density of the fluid,  $r_0$  is the inner radius of the tube and  $J_i(z)$  is the Bessel function of order  $i$  and argument  $z$ .

The root factor method for representing the transcendental expressions in Equation (A) by rational functions may be extended to this and other transfer matrices. The expansion is applied to the transfer matrix directly so viscous effects are included. It is also applicable for an arbitrary terminal condition. The extension follows from the infinite product expansions for the hyperbolic

sine and cosine and Bessel functions. A brief summary of the extension is given below. For details of the method the reader is referred to a study of one of the authors [18].

The infinite product expansions for the sine and cosine hyperbolic and Bessel functions are given by [19]

$$\cosh M(s) = \prod_{n=0}^{\infty} \left[ 1 + \frac{M^2(s)}{(2n+1)^2 \pi^2/4} \right]$$

$$\sinh M(s) = M(s) \prod_{n=1}^{\infty} \left( 1 + \frac{M^2(s)}{(n\pi)^2} \right)$$

$$J_{\nu}(z) = \frac{(z/2)^{\nu}}{\Gamma(\nu+1)} \prod_{n=1}^{\infty} \left( 1 - z^2/\alpha_{\nu,n}^2 \right)$$

where  $J_{\nu}(\alpha_{\nu,n}) = 0$ ,  $n = 1, 2, \dots$  and  $\Gamma(\nu+1)$  is the ordinary gamma function. With the above substitutions  $G(s)$  becomes

$$G(s) = \begin{bmatrix} \prod_{n=0}^{\infty} \left[ 1 + \frac{M^2(s)}{(2n+1)^2 \pi^2/4} \right] & N_1(s) \prod_{n=1}^{\infty} \left( 1 + \frac{M^2(s)}{(n\pi)^2} \right) \\ N_2(s) \prod_{n=1}^{\infty} \left[ 1 + \frac{M^2(s)}{(n\pi)^2} \right] & \prod_{n=0}^{\infty} \left( 1 + \frac{M^2(s)}{(2n+1)^2 \pi^2/4} \right) \end{bmatrix}$$

where

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$$M^2(s) = \frac{8 T_e^2 s}{k} \prod_{n=1}^{\infty} \frac{1 + k s / \alpha_{0,n}^2}{1 + k s / \alpha_{2,n}^2}$$

$$N_1(s) = -\frac{T_e s}{Z_0}$$

$$N_2(s) = -\frac{8 T_e Z_0}{k} \prod_{n=1}^{\infty} \frac{1 + k s / \alpha_{0,n}^2}{1 + k s / \alpha_{2,n}^2}$$

$$k = r^2 \rho / \mu$$

The Bessel function zeros  $\alpha_{j,n}$  monotonically increase with  $n$ , and  $\alpha_{0,n+1}$  approaches  $\alpha_{2,n}$  rapidly with increasing  $n$ . Thus, rational approximations to  $G(s)$  may be readily made by taking only a few terms of the infinite products appearing in  $G(s)$ . A good approximation to the infinite product appearing in  $N_2(s)$  and  $M^2(s)$  for  $|ks| < 400$  is given by

$$\prod_{n=1}^{\infty} \left[ \frac{1 + k s / \alpha_{0,n}^2}{1 + k s / \alpha_{2,n}^2} \right] \approx \frac{(1 + k s / 5.78) (1 + k s / 56.6)}{1 + k s / 40.9}$$

A more accurate representation may be obtained by considering a higher order approximation, but the accuracy obtained with the above expression should suffice for most studies.

Using the approximation above and keeping only one or two terms of the infinite product expansions of the hyperbolic sine and cosine functions in  $G(s)$ , a rational model for the dynamics of a transmission line with "losses"

and arbitrary boundary or terminal conditions results. By keeping only the first two terms of the infinite products, which yields a model of order six in  $s$ , the resulting approximation has been found to give good results for the transient response of a hydraulic line when the only dissipation of energy was due to fluid viscosity.

The infinite product expansion also may be applied to the transfer matrix  $G_{\beta}(s)$  of Equation (24) by expressing the hyperbolic sine and cosine functions as infinite products. The transfer matrix  $G_{\beta}(s)$  results when a linear friction term is added to the momentum equation of water hammer.

In conclusion the infinite product approach may be used to obtain a rational model for a transmission line with "losses" and arbitrary boundary or terminal conditions. The model is readily derived from the transcendental transfer matrix characterizing the line. The approximation resulting from taking only one or two terms of the infinite products is more accurate than a Taylor series expansion. The model is also relatively simple to obtain.

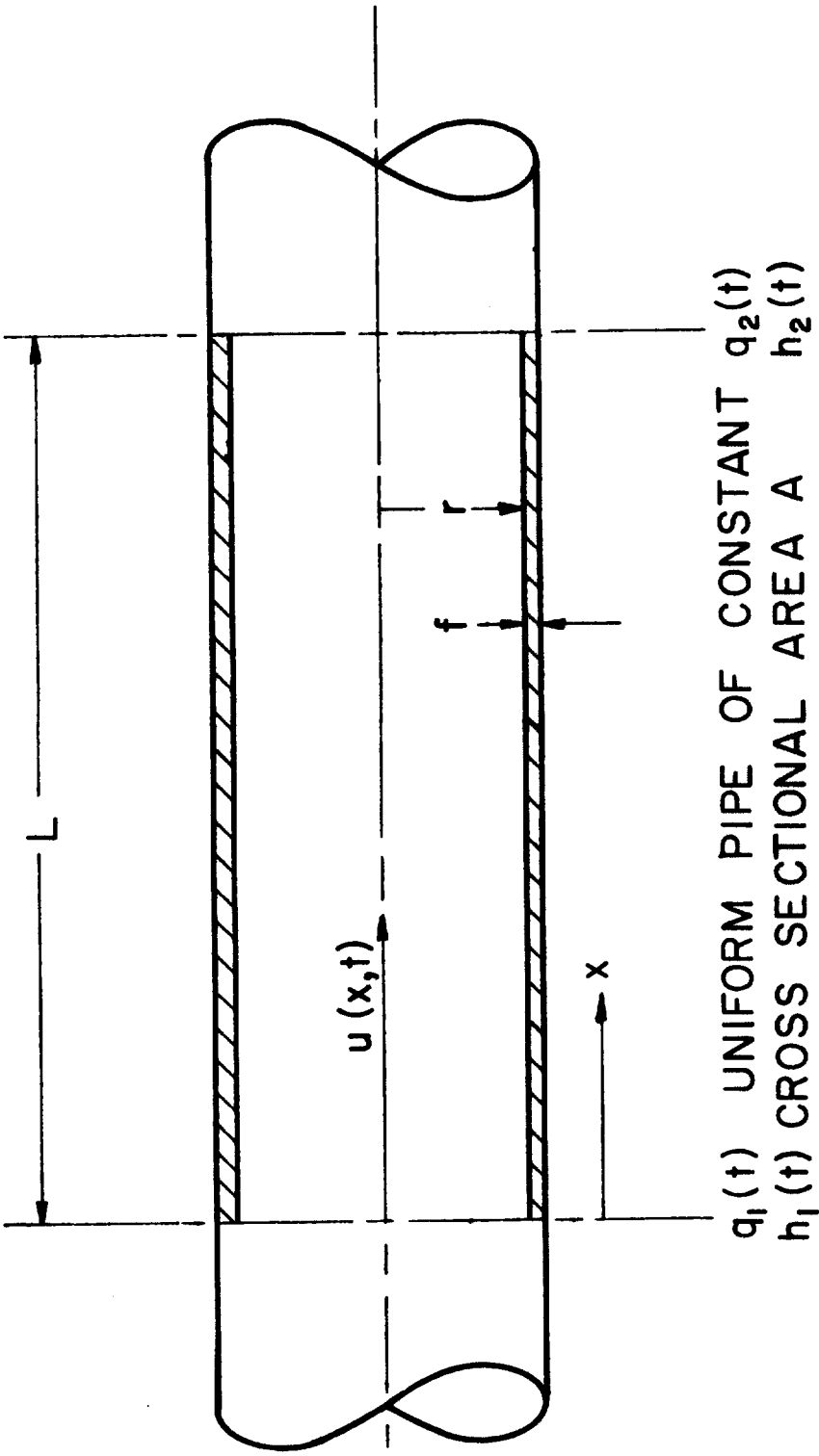
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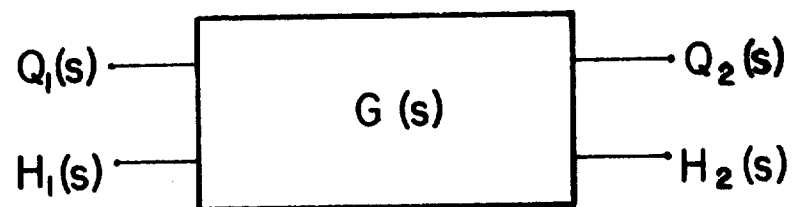
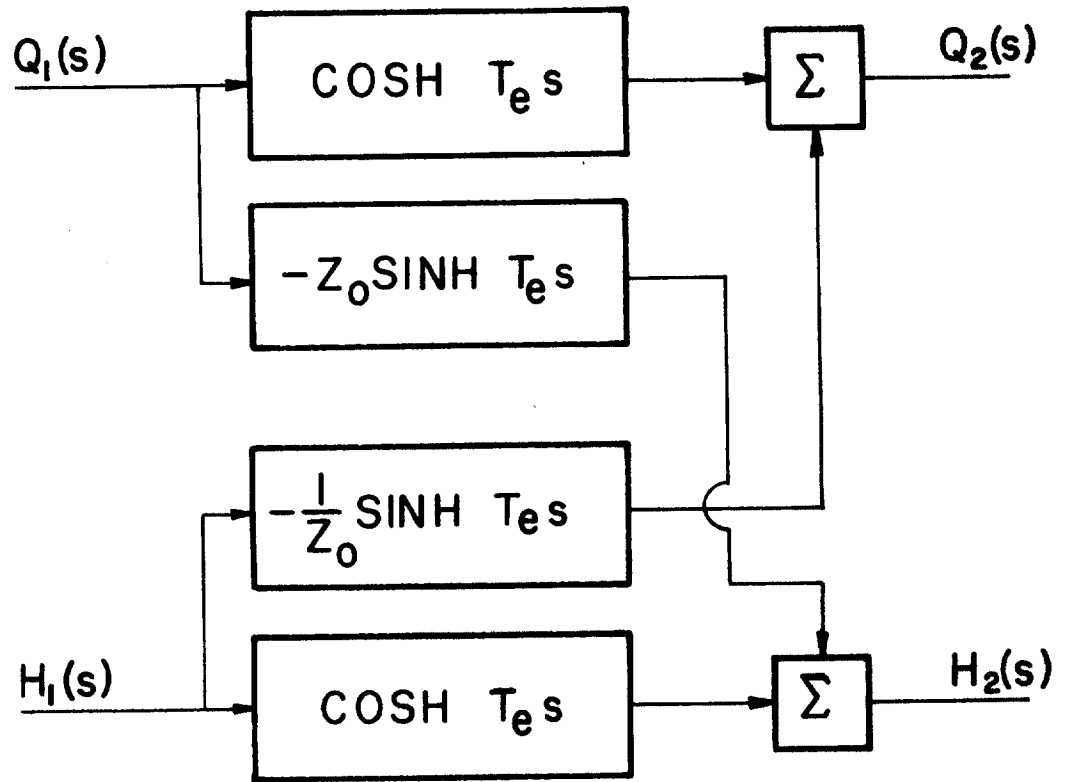
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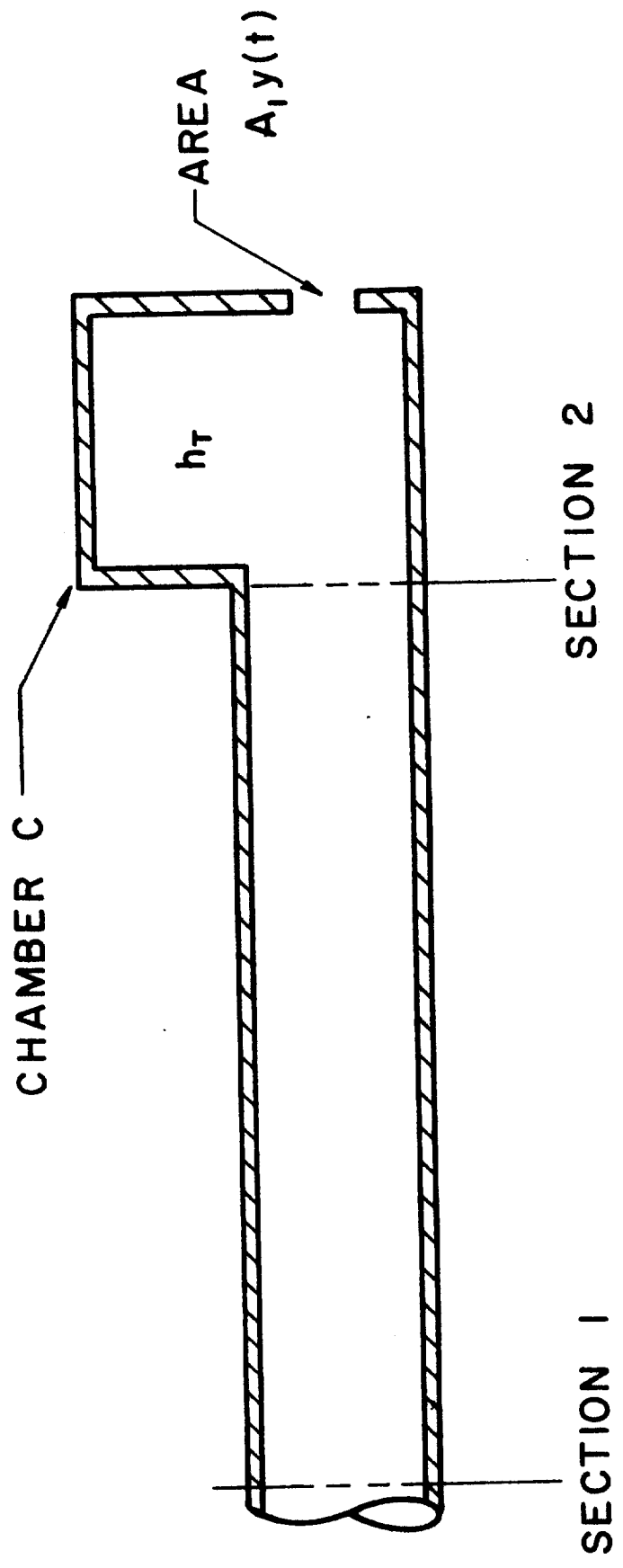
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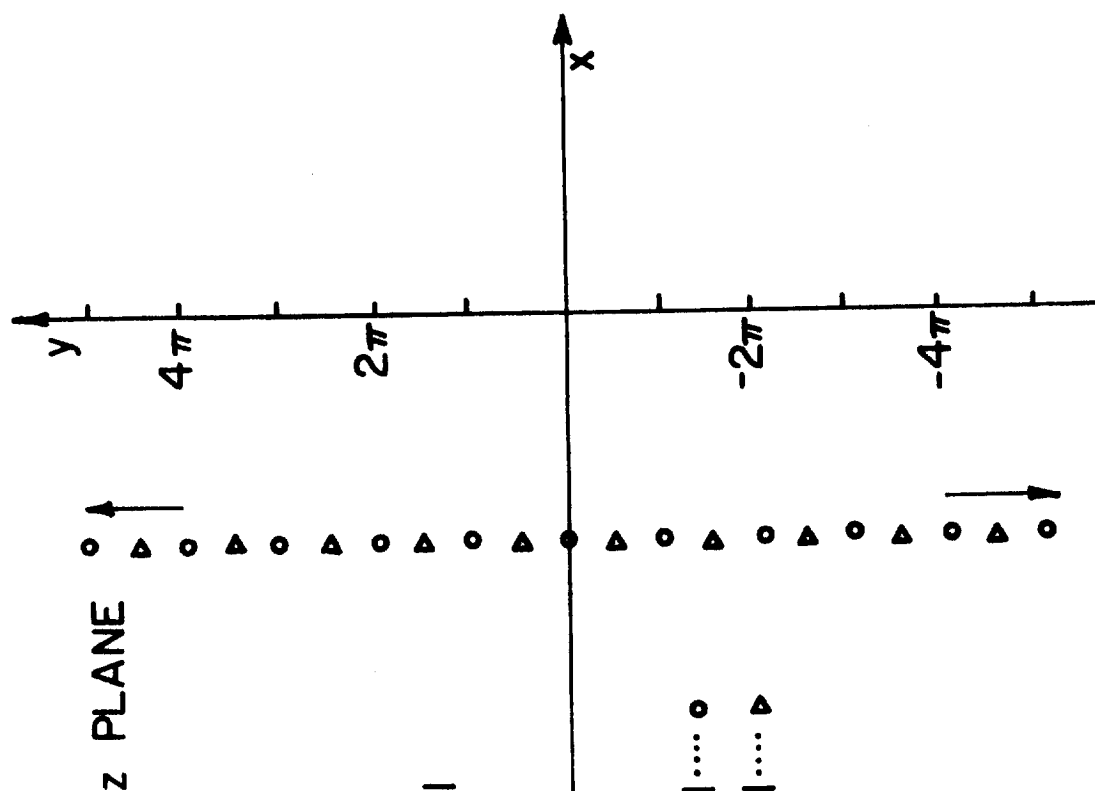
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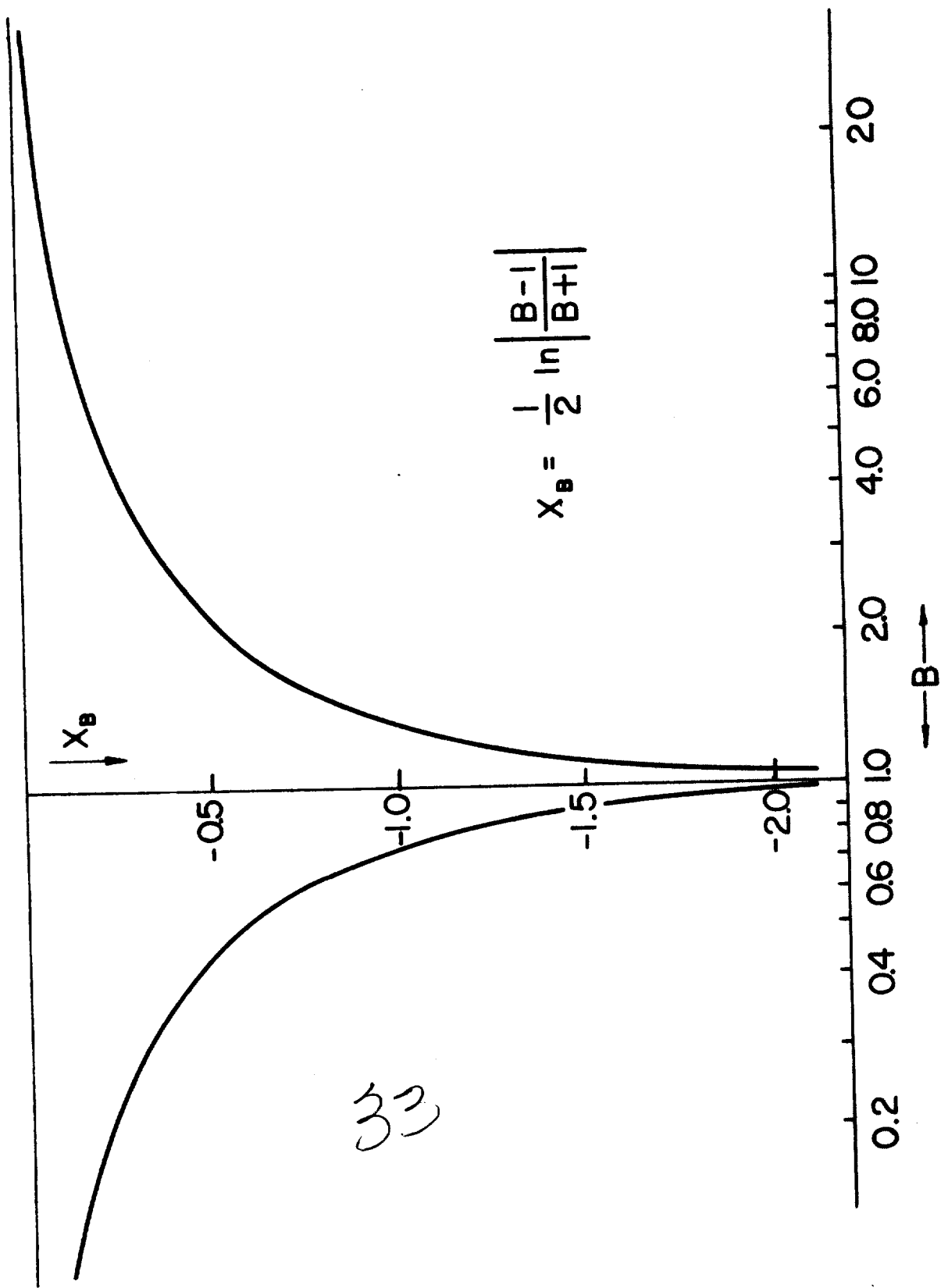




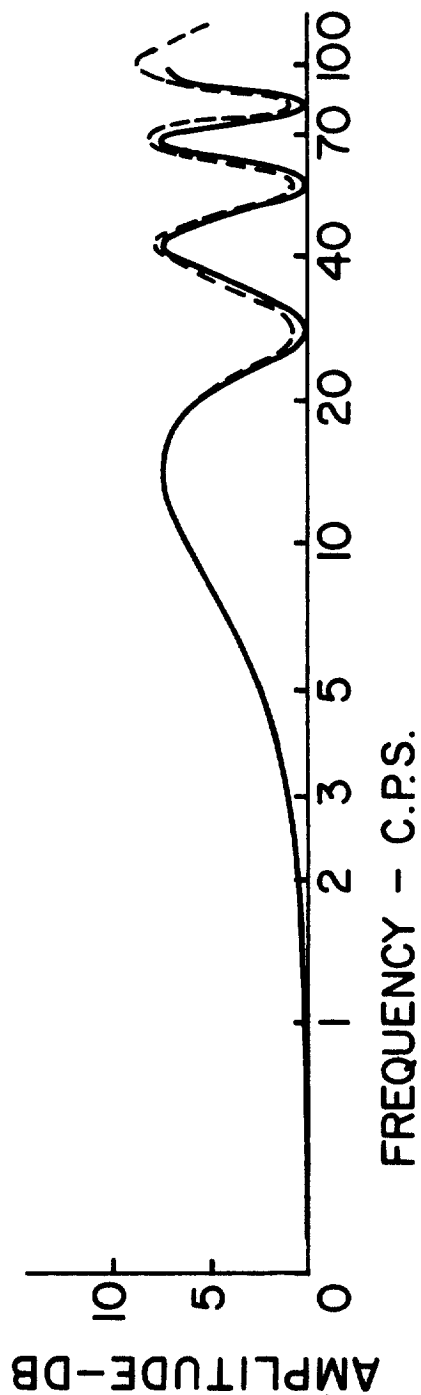
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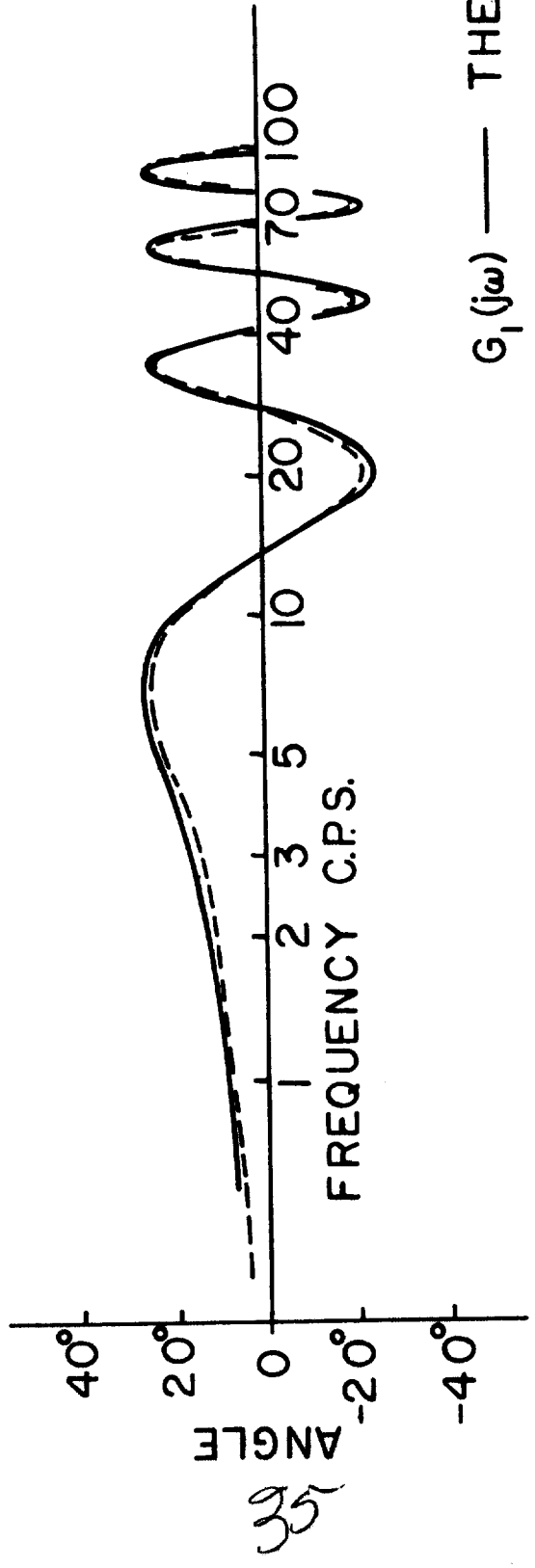


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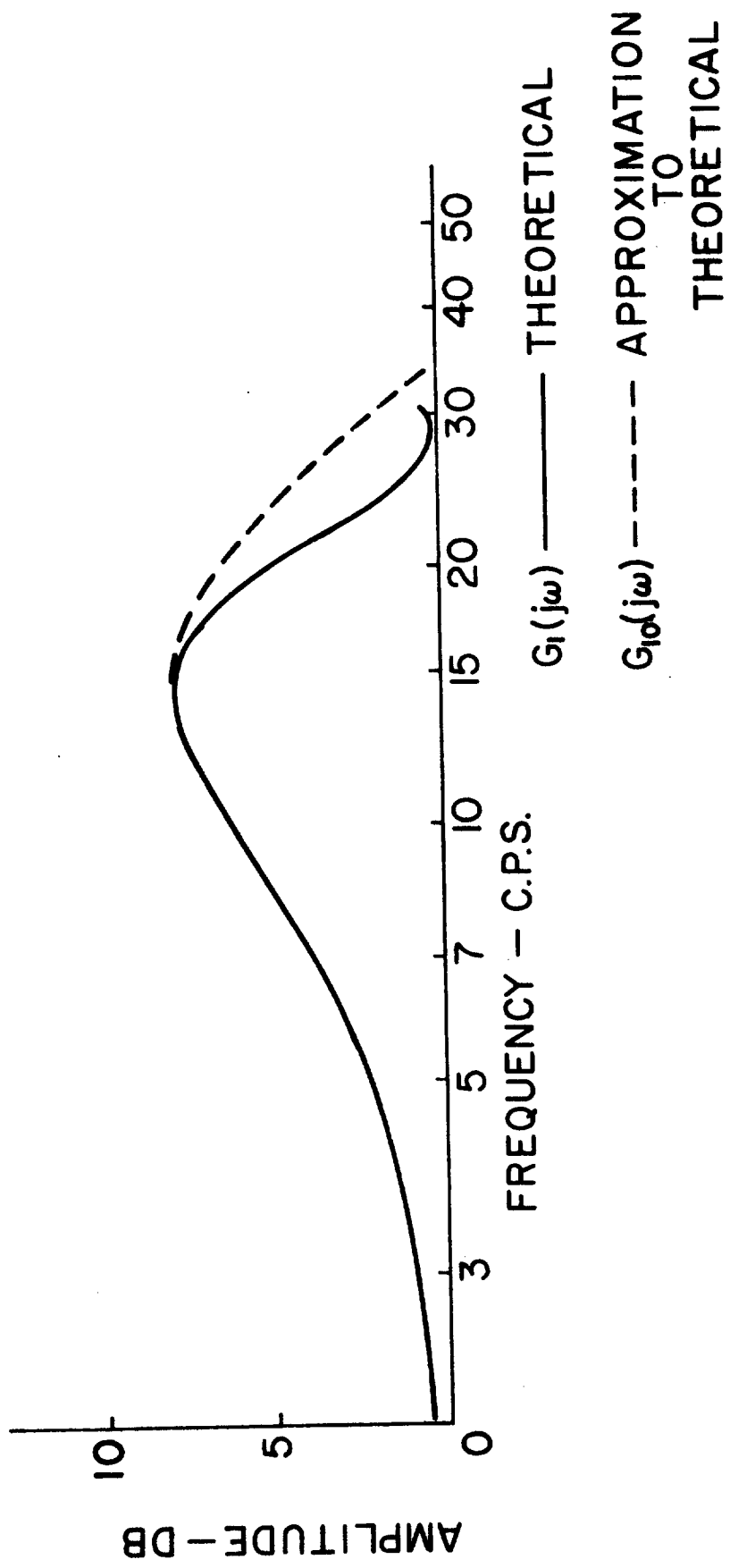
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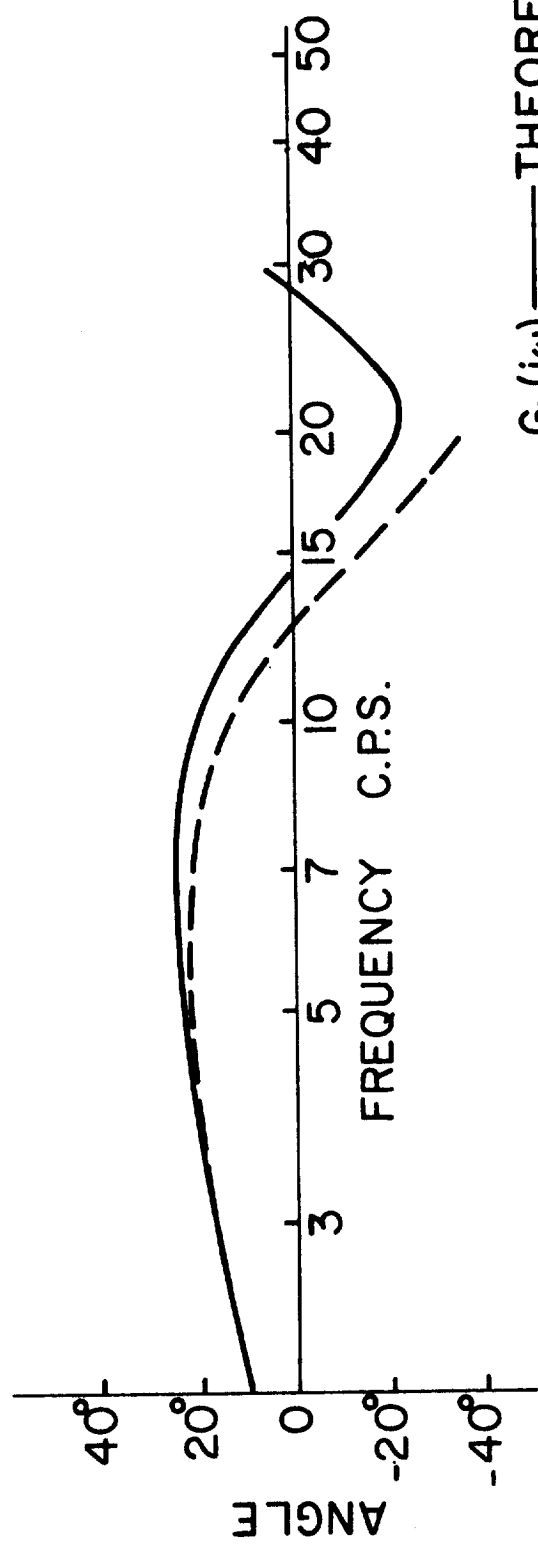


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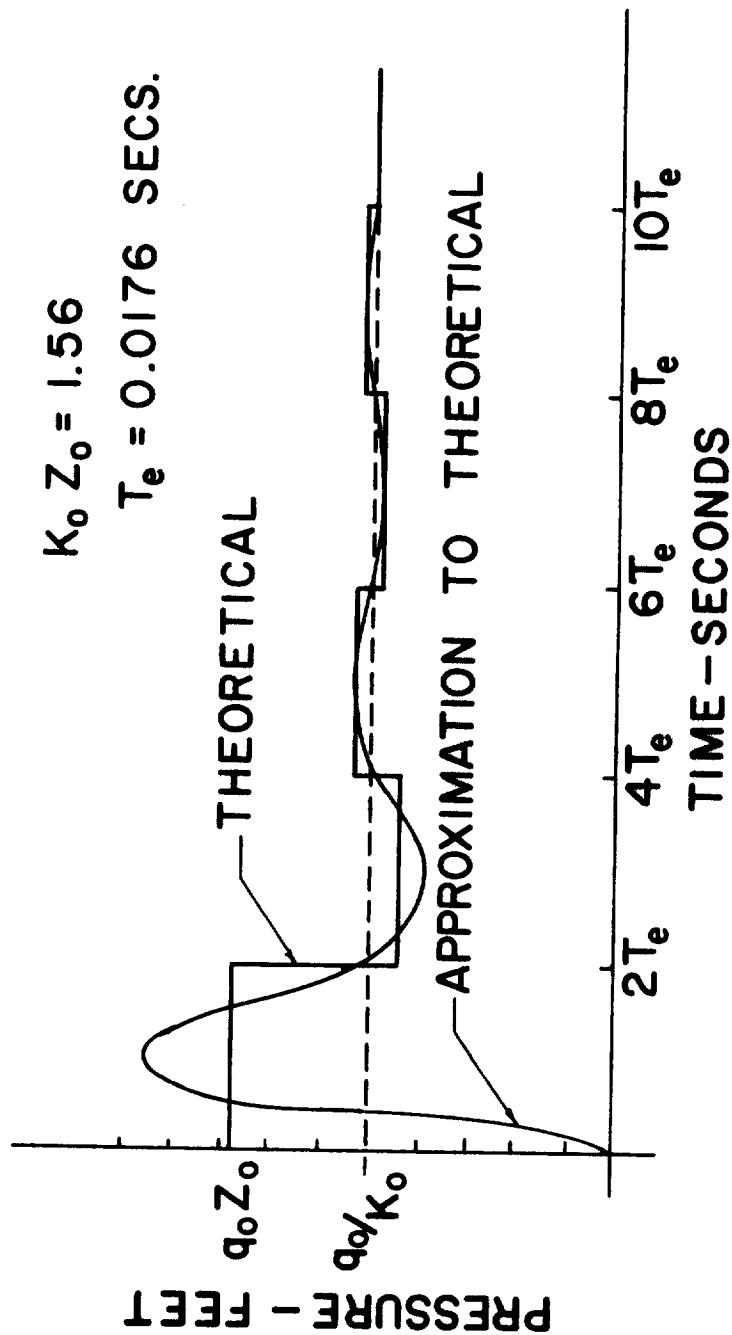


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